Quantum Field Theory

Set 4

Appetizer: Matrix elements

Compute $\langle 0|A_\mu(x)|\epsilon(\vec{k})\rangle$ and $\langle 0|F_{\mu\nu}(x)|\epsilon(\vec{k})\rangle$ where $|\epsilon(\vec{k})\rangle = \epsilon^\mu(\vec{k})a^\dagger_\mu(k)|0\rangle$ is a one-photon state. What is the physical interpretation of these matrix elements ?

Exercise 1: Transformation properties of transverse photons

The polarization of a photon of momentum k_μ is defined by the four vector ε_μ satisfying $\varepsilon_\mu k^\mu = 0$. In the Coulomb gauge we instead use the transverse polarization $\varepsilon^\perp_\mu = (0, \vec{\varepsilon}^\perp)$.

- Check that the conditions $\varepsilon^\perp_0 = 0$ and $\varepsilon^\perp_i k^i = 0$ are *not* Lorentz invariant, which is to say, given a generic polarization vector $\varepsilon^\perp_\mu$ satisfying them, the Lorentz transform $\varepsilon'^\perp_\nu = \Lambda_\mu{}^\nu \varepsilon^\perp_\nu$ has in general $\varepsilon'^\perp_0 \neq 0$ and $\varepsilon'^\perp_i k'^i \neq 0$, where $k'^\mu = \Lambda^\mu{}_\nu k^\nu$.
- Show that it is still possible to find a vector $\tilde{\varepsilon}^\perp_\mu = \varepsilon'^\perp_\mu + \alpha k'_\mu$, i.e. equal to $\varepsilon'^\perp_\mu$ up to a longitudinal component (which is to say, up to a gauge transformation), which satisfies $\tilde{\varepsilon}^\perp_0 = 0$ and $\tilde{\varepsilon}^\perp_i k'^i = 0$.
- Now, working with helicity eigenstates $\varepsilon_\pm(k)$, defined such that $\exp(-iJ\cdot\hat{\mathbf{n}}\,\phi)\varepsilon_\pm(k) = e^{\mp i\phi}\varepsilon_\pm(k)$, where $\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|$, prove that their Lorentz transform will take the following form:

$$\Lambda^\mu{}_\nu\, \varepsilon^\nu_\pm(k) = e^{\mp i\phi(\Lambda,k)}\left(\varepsilon^\mu_\pm(\Lambda k) + \frac{\mp\alpha(\Lambda,k) - i\beta(\Lambda,k)}{\sqrt{2}\omega}(\Lambda k)^\mu\right),$$

  where $\alpha(\Lambda, k)$ and $\beta(\Lambda, k)$ will be functions of the specific Lorentz transform. This shows again that the polarization vectors are not covariant.

Hint: it might be useful to define the following projector onto longitudinal components

$$P_L^{\mu\nu} = \frac{k^\mu \bar{k}^\nu + k^\nu \bar{k}^\mu}{k\cdot\bar{k}}, \qquad \bar{k}^\mu = (k^0, -k^i).$$

Hint 2: For the third question, it might be useful to express the polarization in a reference frame $\varepsilon^\mu_\pm(k) = \Lambda^\mu_{k\ \nu}\,\varepsilon^\nu_\pm(\bar{k})$, where $\bar{k}^\mu = (\omega, 0, 0, \omega)$. Moreover, it should also be useful to recall that any element of the group $ISO(2)$ can be expressed as $W(\alpha,\beta,\phi) = S(\alpha,\beta)R(\phi)$, where R is a rotation and S can be expressed as:

$$S(\alpha,\beta) = \begin{pmatrix} \frac{1}{2}(\alpha^2+\beta^2) & -\beta & \alpha & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\beta & 1 & 0 & \beta \\ \alpha & 0 & 1 & -\alpha \\ \frac{1}{2}(\alpha^2+\beta^2) & -\beta & \alpha & -\frac{1}{2}(\alpha^2+\beta^2) \end{pmatrix}. \tag{1}$$

Exercise 2: Equivalence of Hamiltonian and Lagrangian formalism for massive vectors

Consider the Lagrangian of a massive vector field. Compute the conjugate momenta of A_μ and show that $\Pi_0 = 0$. This means that A_0 is not a dynamical variable but can be expressed in terms of the other fields. Show that $A_0 = -\frac{1}{M^2}\partial_i\Pi^i$. Then show that the Hamiltonian is:

$$H = \int d^3x\left(\frac{1}{2}\Pi^i\Pi^i + \frac{1}{2M^2}(\partial_i\Pi^i)^2 + \frac{1}{4}F^{ij}F^{ij} + \frac{1}{2}M^2A^iA^i\right).$$

Using the following commutation relations

$$\left[A_i(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = i\delta_i^j\delta^3(\vec{x}-\vec{y}),$$
$$[A_i(\vec{x},t),\,A_j(\vec{y},t)] = \left[\Pi^i(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = \left[A_0(\vec{x},t),\,\Pi^j(\vec{y},t)\right] = 0,$$
$$[A_i(\vec{x},t),\,A_0(\vec{y},t)] = -\frac{1}{M^2}\,[A_i(\vec{x},t),\,\partial_m\Pi^m(\vec{y},t)] = \frac{i}{M^2}\partial_i^{(x)}\delta^3(\vec{x}-\vec{y}),$$

show that the Hamilton equations of motion are equivalent to the Lagrange equations of motion.

Homework

Consider the Gupta-Bleuler Lagrangian:

$$\mathcal{L}_{GB} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu).$$

- Compute the angular momentum
$$M^{ij} = J^{ij} + S^{ij},$$
where we separated the orbital and angular part as usual:
$$J^{ij} = \int d^3x J^{0ij}(x), \qquad S^{ij} = \int d^3x S^{0ij}(x),$$
$$J^{\mu\rho\sigma}(x) = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}, \qquad S^{\mu\rho\sigma} = i\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}\left(\mathcal{J}^{\rho\sigma}\right)_\nu^{\ \gamma} A_\gamma.$$
- By working with the algebra of the ladder operators, show that M^{ij} is a physical observable in the sense that:
$$[L, M^{ij}] \propto L.$$
where $L \equiv \partial^\mu A_\mu^-$. Does this happen also for J^{ij} and S^{ij} *separatly*? In other words, is it true that both the orbital and spin angular momentum are physical (measurable) separatly, i.e. $[L, J^{ij}] \propto L$ and $[L, S^{ij}] \propto L$?